Indian Statistical Institute, Bangalore
B. Math. Third Year, Second Semester
Analysis IV

Final Examination Date: 21-04-2016

Maximum marks: 100 Time: 3 hours

Notation: In the following $C_{\mathbb{R}}[0,1]$ is the space of real valued continuous functions on $[0,1]$, with supremum norm.

(1) For $n \geq 1$, define functions s_n in $C_{\mathbb{R}}[0,1]$ by
$$s_n(x) = nx(1-x)^n.$$
Show that the sequence of functions $\{s_n\}$ converge pointwise to a function s on $[0,1]$. Determine as to whether the convergence is uniform. [10]

(2) Let $M = \{f : f \in C_{\mathbb{R}}[0,1], f(0) = 0, f(1) = 1\}$. For f in M, define $\tilde{f}$, by
$$\tilde{f}(x) = \begin{cases} \frac{1}{2}f(3x) & \text{if } 0 \leq x \leq \frac{1}{3} \\ \frac{1}{2} & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{1}{2}(1 + f(3x-2)) & \text{if } \frac{2}{3} < x \leq 1. \end{cases}$$
Show that there exists unique f in M such that $f = \tilde{f}$. [15]

(3) Let D be defined as:
$$D = \{f : f \in C_{\mathbb{R}}[0,1], |f(x) - x| \leq 1 \text{ for all } x\}.$$
Prove or disprove compactness of D. [15]

(4) Let $\mathcal{F}$ be an ultra-filter on the set of natural numbers $\mathbb{N}$. Show that $\mathcal{F}$ contains the co-finite filter $\mathcal{F}_c$ if and only if $\mathcal{F}$ does not contain any finite set. [15]

(5) Let $p(x) = \sum_{n=1}^{\infty} a_n(x - x_0)^n$ be a power series. Take
$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}},$$
with convention $\frac{1}{0} = \infty$, and $\frac{1}{\infty} = 0$. Show that the power series converges absolutely for $|x - x_0| < R$. [15]

(6) Let f, g be two trignometric polynomials:
$$f(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x}; \ g(x) = \sum_{n=-N}^{N} b_n e^{2\pi i n x}.$$
Show that $\widehat{f \star g}(n) = \hat{f}(n)\hat{g}(n)$, for all n. Here $\hat{f}(n)$ are Fourier coefficients of f and $f \star g$ stands for convolution of f and g. [15]

(7) Compute Fourier coefficients for 1-periodic extensions of following functions:

(i) $g(x) = |x - \frac{1}{2}|$ for $0 \leq x < 1$.

(ii)
$$h(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x < 1. \end{cases}$$
[20]